KINEMATIC RELATIONSHIPS AND RELATIVISTIC MOTION

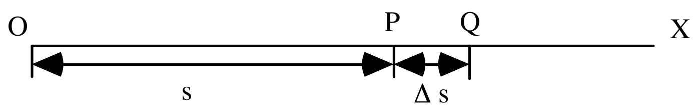
Throughout this course calculus techniques will be used. These techniques are very powerful and a knowledge of integration and differentiation will allow a deeper understanding of the nature of physical phenomena.

Kinematics is the study of the motion of points, making no reference to what causes the motion.

The displacement s of a particle is the length **and** direction from the origin to the particle.

The displacement of the particle is a function of time: $s = f(t)$

Consider a particle moving along OX.



Δ s is covered in time Δt

At time $t + \Delta t$ particle passes Q.

Velocity

average velocity $v_{av} = \dfrac{\Delta s}{\Delta t}$

However the **instantaneous** velocity is different, this is defined as :

$$v = \lim \frac{\Delta s}{\Delta t} \quad (\text{as } \Delta t \to 0) \qquad v = \frac{ds}{dt}$$

Acceleration

velocity changes by Δv in time Δt

$$a_{av} = \frac{\Delta v}{\Delta t}$$

Instantaneous acceleration : $a = \lim \dfrac{\Delta v}{\Delta t}$ (as $\Delta t \to 0$) $\qquad a = \dfrac{dv}{dt}$

$$\text{if} \quad a = \frac{dv}{dt} \quad \text{then} \quad \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2s}{dt^2}$$

$$\boxed{a = \frac{dv}{dt} = \frac{d^2s}{dt^2}}$$

Note: a change in velocity may result from a change in direction (e.g. uniform motion in a circle - see later).

Mathematical Derivation of Equations of Motion for Uniform Acceleration

$$a = \frac{d^2s}{dt^2}$$

Integrate with respect to time:

$$\int \frac{d^2s}{dt^2}\,dt = \int a\,dt$$

$$\frac{ds}{dt} = at + k$$

when $t = 0$ $\frac{ds}{dt} = u$ hence $k = u$

$t = t$ $\frac{ds}{dt} = v$

$$\boldsymbol{v = u + at} \quad ..... \quad \mathbf{1}$$

integrate again : remember that $v = \frac{ds}{dt} = u + at$

$$\int ds = \int u\,dt + \int a\,t\,dt$$

$$s = ut + \frac{1}{2}at^2 + k$$

apply initial conditions: when $t = 0$, $s = 0$ hence $k = 0$

$$\boldsymbol{s = ut + \frac{1}{2}at^2} \quad ..... \quad \mathbf{2}$$

Equations 1 and 2 can now be combined as follows: square both sides of equation 1

$$v^2 = u^2 + 2uat + a^2t^2$$

$$v^2 = u^2 + 2a[ut + \frac{1}{2}at^2]$$

(using equation 2) $$\boldsymbol{v^2 = u^2 + 2as} \quad ..... \quad \mathbf{3}$$

A useful fourth equation is

$$\boldsymbol{s = \frac{(u + v)}{2}t} \quad ..... \quad \mathbf{4}$$

Variable Acceleration

If acceleration depends on time in a simple way, calculus can be used to solve the motion.

Graphs of Motion

The slope or gradient of these graphs provides useful information. Also the area under the graph can have a physical significance.

Displacement - time graphs

$v = \frac{ds}{dt}$ slope = instantaneous velocity.

Area under graph - no meaning.

Velocity - time graphs

$a = \frac{dv}{dt}$ slope = instantaneous acceleration.

Also $s = \int v \, dt$ Area under v-t graph gives the displacement.

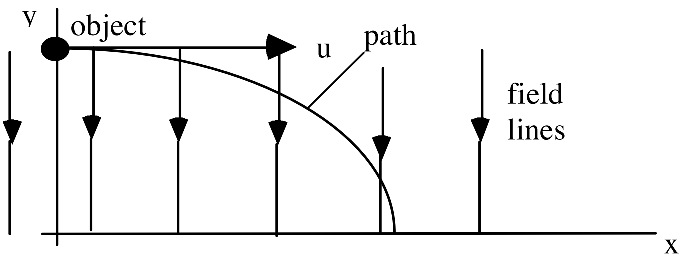
Calculations Involving Uniform Accelerations

Examples of **uniform** acceleration are:

- vertical motion of a projectile near the Earth's surface, where the acceleration is $g = 9.8 \text{ m s}^{-2}$ vertically downwards
- rectilinear (i.e. straight line) motion e.g. vehicle accelerating along a road.

These have been covered previously; however a fuller mathematical treatment for projectiles is appropriate at this level.

Consider the simple case of an object projected with an initial velocity u at right angles to the Earth's gravitational field - (locally the field lines may be considered parallel).



$a = g$, time to travel distance x across field = t

$$t = \frac{x}{u}$$

apply $y = u_y t + \frac{1}{2} a t^2$, $u_y t = 0$ and $a = g$

$$y = \frac{1}{2} \cdot g \cdot \frac{x^2}{u^2}$$

$$y = \left[\frac{1}{2} \cdot \frac{g}{u^2} \right] . x^2$$

Now g and u are constants, $y \propto x^2$ and we have the equation of a **parabola**.

The above proof and equations are **not** required for examination purposes.

ANGULAR MOTION

The angular velocity of a rotating body is defined as the rate of change of angular displacement.

$$\omega = \frac{d\theta}{dt}$$

where ω (rad s^{-1}) is the angular velocity
θ (rad) is the angular displacement

The radian (rad) is a unit of angle: $180° = \pi$ rad

Angular acceleration, $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ unit of α: rad s^{-2}

We assume for this course that α is **constant**.

The derivation of the equations for angular motion are very similar to those for linear motion seen earlier. The derivations of these equations are **not required** for examination purposes.

Angular Motion Relationships

$$\omega = \omega_o + \alpha t \quad ..... \quad 1$$

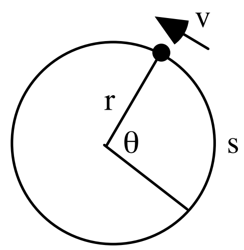
$$\theta = \omega_o t + \frac{1}{2}\alpha t^2 \quad ..... \quad 2$$

$$\omega^2 = \omega_o^2 + 2\alpha\theta \quad ..... \quad 3$$

You will note that these **angular** equations have exactly the same form as the **linear** equations.
Remember that these equations only apply for **uniform** angular accelerations.

Uniform Motion in a Circle

Consider a particle moving with uniform speed in a circular path as shown opposite.



$$\omega = \frac{d\theta}{dt}$$

The rotational speed v is constant, ω is also constant.
T is the period of the motion and is the time taken to cover 2π radians.

$$\omega = \frac{2\pi}{T} \quad \text{but} \quad v = \frac{2\pi r}{T}$$

$$v = r\omega$$

(Note: s is the arc swept out by the particle and $s = r\theta$)

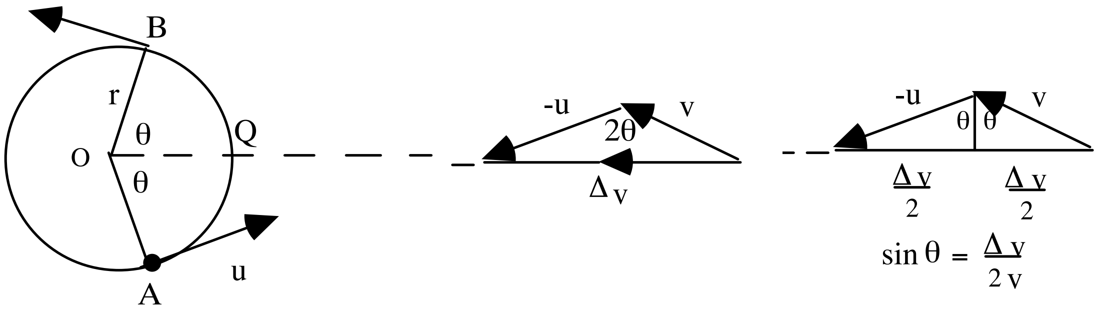
Radial and angular acceleration

The angular acceleration $\alpha = \frac{d\omega}{dt}$ and the radial acceleration $a = \frac{dv}{dt}$.

Therefore since $v = r\omega$, then at any instant $\frac{dv}{dt} = r\frac{d\omega}{dt}$ giving

$$\boxed{a = r\alpha}$$

Radial Acceleration



The particle travels from A to B in time Δt and with speed v, thus $|u| = |v|$ and $\Delta v = v + (-u)$ which is $\Delta v = v - u$

$$\Delta t = \frac{\text{arc } AB}{v} = \frac{r(2\theta)}{v}$$

average acceleration, $a_{av} = \frac{\Delta v}{\Delta t} = \frac{2v\sin\theta}{\Delta t}$

$$= \frac{2v\sin\theta}{r\,2\theta / v} = \frac{v^2}{r}\cdot\frac{\sin\theta}{\theta}$$

As $\theta \to 0$, $a_{av} \to$ instantaneous acceleration at point Q:

$$a = \frac{v^2}{r}\cdot\left[\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right] \quad \text{but} \left[\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right] = 1$$

when θ is small and is measured in radians $\sin\theta = \theta$.

$$\boxed{a = \frac{v^2}{r} = \omega^2 r} \quad \text{since } v = r\omega$$

The **direction** of this acceleration is always towards the **centre** of the circle.

Note: This is **not** a uniform acceleration. Radial acceleration is fixed only in size. Compare this with the angular acceleration which is constant for problems in this course.

This motion is typical of many **central force** type motions e.g. Planetary Motion, electrons 'orbiting' nuclei and electrons injected at right angles to a uniform magnetic field which will be covered later in the course.

Thus any object performing circular orbits at uniform speed must have a **centre-seeking** or **central** force responsible for the motion.

Central Force

Does a rotating body really have an inward acceleration (and hence an inward force)?

Argument Most people have experienced the sensation of being in a car or a bus which is turning a corner at high speed. The feeling of being 'thrown to the outside of the curve' is very strong, especially if you slide along the seat. What happens here is that the friction between yourself and the seat is insufficient to provide the central force needed to deviate you from the straight line path you were following before the turn. In fact, instead of being thrown outwards, you are, in reality, continuing in a straight line while the car moves inwards. Eventually you are moved from the straight line path by the inward (central) force provided by the door.

Magnitude of the Force

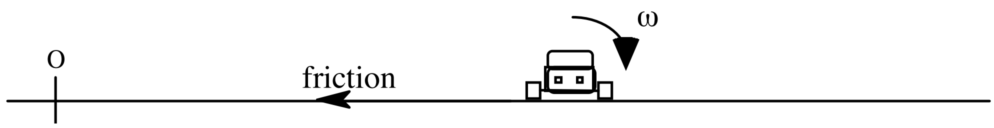
$$F = m a \qquad \text{but } a = \frac{v^2}{r} \quad \text{or} \quad a = \omega^2 r$$

Thus central force, $\boxed{F = m\frac{v^2}{r} \quad \text{or} \quad F = m r \omega^2}$ since $v = r\omega$

Examples

1. A Car on a Flat Track

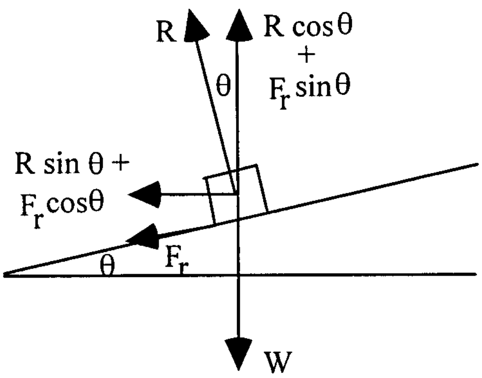
If the car goes too fast, the car 'breaks away' at a tangent. The force of friction is not enough to supply an adequate central force.



2. A Car on a Banked Track

For tracks of similar surface properties, a car will be able to go faster on a banked track before going off at a tangent because there is a component of the normal reaction as well as a component of friction, F_r, supplying the central force.

The central force is $R \sin\theta + F_r \cos\theta$ which reduces to $R \sin\theta$ when the friction is zero. The analysis on the right hand side is for the friction F_r equal to **zero**.



R is the 'normal reaction' force of the track on the car.

In the vertical direction there is no acceleration:

$$R \cos\theta = mg \quad \text{........ 1}$$

In the radial direction there is a central acceleration:

$$R \sin\theta = \frac{mv^2}{r} \quad \text{........ 2}$$

Divide Eq. 2 by Eq. 1:

$$\tan\theta = \frac{v^2}{gr} \qquad \text{(assumes friction is zero)}$$

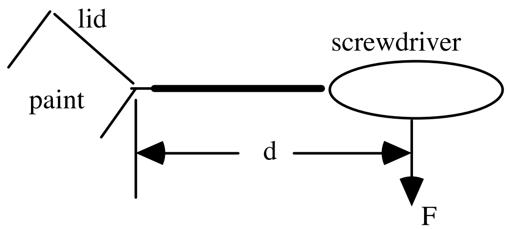
(This equation applies to all cases of 'banking' including aircraft turning in horizontal circles)

ROTATIONAL DYNAMICS

Moment of a force

The **moment** of a force is the **turning effect** it can produce.
Examples of moments are:

using a long handled screwdriver to 'lever off' the lid of a paint tin, see below
using a claw hammer to remove a nail from a block of wood or levering off a cap from a bottle.

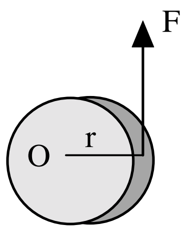


The magnitude of the moment of the force (or the turning effect) is F x d.

Where F is the force and d is the perpendicular distance from the direction of the force to the turning point.

Torque

For cases where a force is applied and this causes rotation about an axis, the moment of the force can be termed the **torque**.



Consider a force F applied tangentially to the rim of a disc which can rotate about an axis O through its centre. The radius of the circle is r.
The torque T associated with this force F is defined to be the force multiplied by the radius r.

$T = F \times r$	unit of T: newton metre (N m)

Torque is a **vector** quantity. The direction of the torque vector is at right angles to the plane containing both r and F and lies along the axis of rotation. (For interest only, in the example shown in the diagram torque, T, points out of the page).

A force acting on the rim of an object will cause the object to rotate; e.g. applying a push or a pull force to a door to open and close , providing it creates a non-zero resulting torque. The distance from the axis of rotation is an important measurement when calculating torque. It is instructive to measure the relative forces required to open a door by pulling with a spring balance firstly at the handle and then pulling in the middle of the door. Another example would be a **torque wrench** which is used to rotate the wheel nuts on a car to a certain 'tightness' as specified by the manufacturer.

An **unbalanced torque** will produce an **angular acceleration**. In the above diagram if there are no other forces then the force F will cause the object to rotate.

Inertia

In linear dynamics an unbalanced force produces a linear acceleration. The magnitude of the linear acceleration produced by a given unbalanced force will depend on the mass of the object, that is on its inertia. The word inertia can be loosely described as 'resistance to change in motion of an object' Objects with a large mass are difficult to start moving and once moving are difficult to stop.

Moment of Inertia

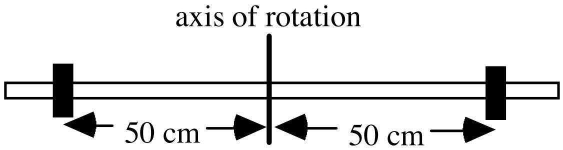
The moment of inertia I of an object can be described as its resistance to change in its angular motion. The moment of inertia I for rotational motion is analogous to the mass m for linear motion

The moment of inertia I of an object depends on the mass **and** the distribution of the mass about the axis of rotation.

For a mass m at a distance r from the axis of rotation the moment of inertia of this mass is given by the mass m multiplied by r^2.

$$\boxed{I = m r^2} \quad \text{unit of I: kg m}^2$$

For example, a very light rod has two 0.8 kg masses each at a distance of 50 cm from the axis of rotation.



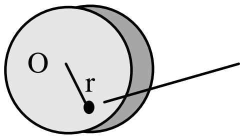
The moment of inertia of each mass is $m r^2 = 0.8 \times 0.5^2 = 0.2$ kg m^2 giving a total moment of inertia $I = 0.4$ kg m^2. Notice that we assume that all the mass is at the 50 cm distance. The small moment of inertia of the light rod has been ignored.

Another example is a hoop, with very light spokes connecting the hoop to an axis of rotation through the centre of the hoop and perpendicular to the plane of the hoop, e.g. a bicycle wheel. Almost all the mass of the hoop is at a distance R, where R is the radius of the hoop. Hence $I = M R^2$ where M is the total mass of the hoop.

For objects where all the mass can be considered to be at the **same** distance from the axis of rotation this equation $I = m r^2$ can be used directly.

However most objects do **not** have all their mass at a single distance from the axis of rotation and we must consider the distribution of the mass.

Moment of inertia and mass distribution



Consider a small particle of the disc as shown. This particle of mass m is at a distance r from the axis of rotation 0.

The contribution of this mass to the moment of inertia of the whole object (in this case a disc) is given by the mass m multiplied by r^2. To obtain the moment of inertia of the disc we need to consider all the particles of the disc, each at their different distances.

Any object can be considered to be made of n particles each of mass m. Each particle is at a particular radius r from the axis of rotation. The moment of inertia of the object is determined by the summation of all these n particles e.g. $\sum (m r^2)$. Calculus methods are used to determine the moments of inertia of extended objects. In this course, moments of inertia of extended objects, about specific axes, will be given.

It can be shown that the moment of inertia of a uniform rod of length L and total mass M through its centre is $\frac{ML^2}{12}$, but the moment of inertia of the same rod through its end is $\frac{ML^2}{3}$, i.e. four times bigger. This is because it is harder to make the rod rotate about an axis at the end than an axis through its middle because there are now more particles at a greater distance from the axis of rotation.

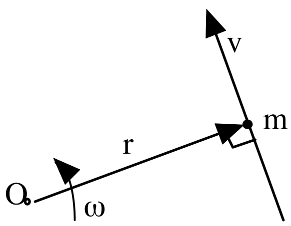
Torque and Moment of Inertia

An **unbalanced torque** will produce an **angular acceleration**. As discussed above, the moment of inertia of an object is the opposition to a change in its angular motion. Thus the angular acceleration α produced by a given torque T will depend on the moment of inertia I of that object.

$$T = I \alpha$$

Angular Momentum

The angular momentum L of a particle about an axis is defined as the **moment** of momentum.



A particle of mass m rotates at ω rad s^{-1} about the point O.
The linear momentum p = m v.
The moment of p = m v r (r is perpendicular to v).

Thus the angular momentum of this particle $= m v r = m r^2 \omega$, since $v = r \omega$.

For a rigid object about a fixed axis the angular momentum L will be the summation of all the individual angular momenta. Thus the angular momentum L of an object is given by $\sum (m\, r^2\, \omega)$. This can be written as $\omega \sum (m\, r^2)$ since all the individual parts of the object will have the same angular velocity ω. Also we have $I = \sum (m\, r^2)$.

Thus the angular momentum of a rigid body is:

$$\boxed{L = I\, \omega} \qquad \text{unit of L: kg m}^2\ \text{s}^{-1}.$$

Notice that the angular momentum of a rigid object about a fixed axis **depends** on the moment of inertia.

Angular momentum is a **vector** quantity. The **direction** of this vector is at right angles to the plane containing v (since p = m v and mass is scalar) and r and lies along the axis of rotation. For interest only, in the above example L is out of the page. (Consideration of the vector nature of T and L will not be required for assessment purposes.)

Conservation of angular momentum

The **total** angular momentum before an impact will equal the **total** angular momentum after impact providing no external torques are acting.

You will meet a variety of problems which involve use of the conservation of angular momentum during collisions for their solution.

Rotational Kinetic Energy

The rotational kinetic energy of a rigid object also depends on the moment of inertia. For an object of moment of inertia I rotating uniformly at ω rad s^{-1} the rotational kinetic energy is given by:

$$\boxed{E_k = \frac{1}{2} I\, \omega^2}$$

Energy and work done

If a torque T is applied through an angular displacement θ, then the work done $= T\,\theta$.
Doing work produces a transfer of energy, $T\,\theta = I\,\omega^2 - I\,\omega_0^2$ (work done = ΔE_k).

Summary and Comparison of Linear and Angular Equations

Quantity	*Linear Motion*		*Angular Motion*
acceleration	a	$(a = r\alpha)$	α
velocity	$v = u + at$	$(v = r\omega)$	$\omega = \omega_o + \alpha t$
displacement	$s = ut + \frac{1}{2}at^2$		$\theta = \omega_o t + \frac{1}{2}\alpha t^2$
momentum	$p = mv$		$L = I\omega$
kinetic energy	$\frac{1}{2}mv^2$		$\frac{1}{2}I\omega^2$
Newton's 2nd law	$F = m\frac{dv}{dt} = ma$		$T = I\frac{d\omega}{dt} = I\alpha$

Laws

Conservation of momentum	$m_A u_A + m_B u_B = m_A v_A + m_B v_B$	$I_A\omega_{oA} + I_B\omega_{oB} = I_A\omega_A + I_B\omega_B$
Conservation of energy	$F.s = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$	$T\theta = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_o^2$

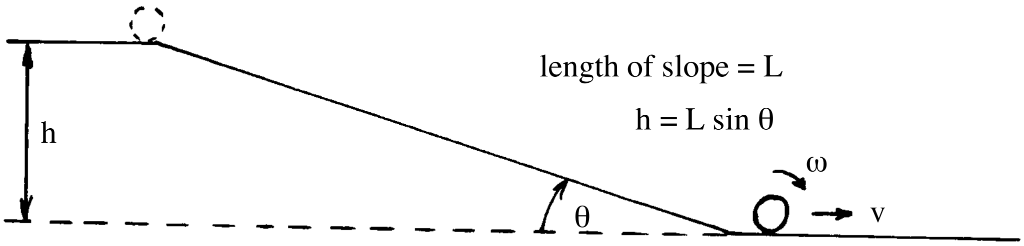
Some Moments of Inertia (for reference)

Thin disc about an axis through its centre and perpendicular to the disc.	$I = \frac{1}{2}MR^2$	R = radius of disc
Thin rod about its centre	$I = \frac{1}{12}ML^2$	L = length of rod
Thin hoop about its centre	$I = MR^2$	R = radius of hoop
Sphere about its centre	$I = \frac{2}{5}MR^2$	R = radius of sphere

Where M is the total mass of the object in each case.

Objects Rolling down an Inclined Plane

When an object such as a sphere or cylinder is allowed to run down a slope, the E_p at the top, (m g h), will be converted to both **linear** ($\frac{1}{2}$ m v^2) and **angular** ($\frac{1}{2}$ I ω^2) kinetic energy.



An equation for the energy of the motion (assume no slipping) is given below.

$$m\,g\,h = \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2$$

The above formula can be used in an experimental determination of the moment of inertia of a circular object.

Example

A solid cylinder is allowed to roll from rest down a shallow slope of length 2.0 m. When the height of the slope is 0.02 m, the time taken to roll down the slope is 7.8 s. The mass of the cylinder is 10 kg and its radius is 0.10 m.

Using this information about the motion of the cylinder and the equation above, calculate the moment of inertia of the cylinder.

Solution

$$E_P = m\,g\,h = 10 \times 9.8 \times 0.02 = 1.96 \text{ J}$$

Change in gravitational E_p = change in linear E_k + change in rotational E_k

$$m\,g\,h = (\frac{1}{2} m v^2 - 0) + (\frac{1}{2} I \omega^2 - 0)$$

$$s = \frac{(u + v)}{2} t$$

$$2.0 = \frac{(0 + v)}{2} \times 7.8$$

$$v = \frac{4.0}{7.8} = 0.513 \text{ m s}^{-1}$$

$$\omega = \frac{v}{r} = \frac{0.513}{0.10} = 5.13 \text{ rad s}^{-1}$$

$$E_{k(lin)} = \frac{1}{2} m v^2 = \frac{1}{2} \times 10 (0.513)^2 = 1.32 \text{ J}$$

$$E_{k(rot)} = E_P - E_{k(lin)}$$

$$\frac{1}{2} I \omega^2 = 1.96 - 1.32 = 0.64 \text{ J}$$

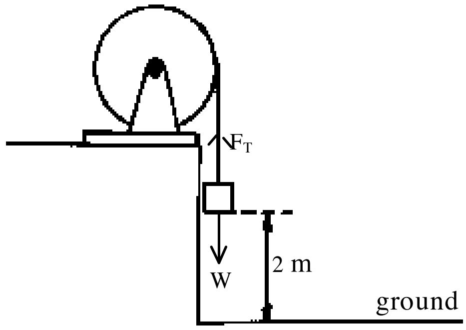
$$I = \frac{2 \times 0.64}{\omega^2} = \frac{2 \times 0.64}{(5.13)^2}$$

$$I = 0.049 \text{ kg m}^2$$

The Flywheel

Example

The flywheel shown below comprises a solid cylinder mounted through its centre and free to rotate in the vertical plane.



Flywheel: mass = 25 kg
radius = 0.30 m.

Mass of hanging weight = 2.5 kg

The hanging weight is released. This results in an angular acceleration of the flywheel. Assume that the effects of friction are negligible.

(a) Calculate the angular acceleration of the flywheel.

(b) Calculate the angular velocity of the flywheel just as the weight reaches ground level.

Solution

(a) We need to know I, the moment of inertia of the flywheel: $I = \frac{1}{2} M R^2$

$$I = \frac{1}{2} \times 25 \times (0.30)^2 = 1.125 \text{ kg m}^2$$

Consider the forces acting on the flywheel: $W - F_T = m a$ where $m = 2.5$ kg

$$24.5 - F_T = 2.5 \times 0.30\ \alpha \quad (a = r\alpha)$$

$$F_T = 24.5 - 0.75\ \alpha$$

$$\text{Torque, } T = F_T \times r = (24.5 - 0.75\ \alpha) \times 0.30$$

and $T = I\alpha = 1.125\ \alpha$

Thus $1.125\ \alpha = 7.35 - 0.225\ \alpha$

$$\alpha = \frac{7.35}{1.35} = 5.44 \text{ rad s}^{-2}$$

(b) To calculate the angular velocity we will need to know θ, the angular displacement for a length of rope 2.0 m long being unwound.

$$\text{circumference} = 2\pi r = 2\pi \times 0.30$$

$$\text{no. of revs} = \frac{\text{length unwound}}{\text{circumference}} = \frac{2.0 \text{ m}}{2\pi \times 0.30 \text{ m}}$$

$$\theta = 2\pi \times \text{no. of revs} = 2\pi \times \frac{2.0}{2\pi \times 0.30} = 6.67 \text{ rad}$$

$\omega_o = 0$	apply $\omega^2 = \omega_o^2 + 2\alpha\theta$
$\omega = ?$	$\omega^2 = 0 + 2 \times 5.44 \times 6.67$
$\alpha = 5.44$ rad s^{-2}	$\omega^2 = 72.57$
$\theta = 6.67$ rad	$\omega = 8.52$ rad s^{-1}

Frictional Torque

Example

The friction acting at the axle of a bicycle wheel can be investigated as follows. The wheel, of mass 1.2 kg and radius 0.50 m, is mounted so that it is free to rotate in the vertical plane. A driving torque is applied and when the wheel is rotating at 5.0 revs per second the driving torque is removed. The wheel then takes 2.0 minutes to stop.

(a) Assuming that all the spokes of the wheel are very light and the radius of the wheel is 0.50 m, calculate the moment of inertia of the wheel.

(b) Calculate the frictional torque which causes the wheel to come to rest.

(c) The effective radius of the axle is 1.5 cm. Calculate the force of friction acting at the axle.

(d) Calculate the kinetic energy lost by the wheel. Where has this energy gone?

Solution

(a) In this case I for wheel $= MR^2$

$$I = 1.2 \times (0.50)^2 \quad (M = 1.2\text{ kg}, \; R = 0.50\text{ m})$$

$$I = 0.30\text{ kg m}^2$$

(b) To find frictional torque we need the angular acceleration (α), because $T = I\alpha$

$\omega = 0, t = 120$ s $\qquad \alpha = \dfrac{\omega - \omega_o}{t}$

$\omega_o = 5.0$ r.p.s. $\qquad = \dfrac{0 - 31.4}{120}$

$= 31.4\text{ rad s}^{-1}$ $\qquad \alpha = -0.262\text{ rad s}^{-2}$

Now use $\qquad T = I\alpha$

$$= 0.3 \times (-0.262)$$

$$T = -0.0786\text{ N m}$$

(c) Also $\qquad T = rF \qquad (r = 1.5\text{ cm} = 0.015\text{ m})$

$$F = \frac{T}{r} = -\frac{0.0786}{0.015} = -5.24\text{ N}$$

i.e. negative value indicates force *opposing* motion.

(d)

$$E_{k(rot)} = \frac{1}{2} I \omega_o^2$$

$$= \frac{1}{2} \times 0.30 \times (31.4)^2$$

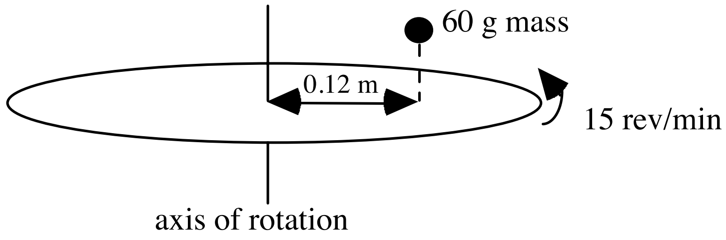
$$= 148\text{ J}$$

When the wheel stops $E_{k(rot)} = 0$. This 148 J will have changed to heat in the axle due to the work done by the force of friction.

Conservation of Angular Momentum

Example

A turntable, which is rotating on frictionless bearings, rotates at an angular speed of 15 revolutions per minute. A mass of 60 g is dropped from rest just above the disc at a distance of 0.12 m from the axis of rotation through its centre.



As a result of this impact, it is observed that the rate of rotation of the disc is reduced to 10 revolutions per minute.

(a) Use this information and the principle of conservation of angular momentum to calculate the moment of inertia of the disc.

(b) Show by calculation whether this is an elastic or inelastic collision.

Solution

(a) Moment of inertia of disc $= I$

Moment of inertia of 60 g mass $= m r^2$ (treat as 'particle' at radius r)

$= 0.06 \times (0.12)^2$

$I_{mass} = 8.64 \times 10^{-4}$ kg m^2

initial angular velocity $= \omega_o = 15$ rev min^{-1} $= \dfrac{15 \times 2\pi}{60}$

$\omega_o = 1.57$ rad s^{-1}

final angular velocity $= \omega = 10$ rev min^{-1}

$= 1.05$ rad s^{-1}

Total angular momentum before impact = total angular momentum after impact

$$I\,\omega_o = (I + I_{mass})\,\omega$$

$$I \times 1.57 = (I + 8.64 \times 10^{-4}) \times 1.05$$

$$0.52\,I = 9.072 \times 10^{-4}$$

$$I = \frac{9.072 \times 10^{-4}}{0.52} = 1.74 \times 10^{-3} \text{ kg m}^2$$

(b) E_k before impact $= \frac{1}{2} I \omega_o^2 = \frac{1}{2} \times 1.74 \times 10^{-3} \times (1.57)^2 = 2.14 \times 10^{-3}$ J

E_k after impact $= \frac{1}{2}(I + I_{mass})\,\omega^2 = \frac{1}{2} \times 2.60 \times 10^{-3} \times (1.05)^2 = 1.43 \times 10^{-3}$ J

E_k difference $= 7.1 \times 10^{-4}$ J

Thus the collision is **inelastic**. The energy difference will be changed to heat.

GRAVITATION

Historical Introduction

The development of what we know about the Earth, Solar System and Universe is a fascinating study in its own right. From earliest times Man has wondered at and speculated over the 'Nature of the Heavens'. It is hardly surprising that most people (until around 1500 A.D.) thought that the Sun revolved around the Earth because that is what it seems to do! Similarly most people were sure that the Earth was flat until there was definite proof from sailors who had ventured round the world and not fallen off!

It may prove useful therefore to give a brief historical introduction so that we may set this topic in perspective. For the interested student, you are referred to a most readable account of Gravitation which appears in "Physics for the Inquiring Mind" by Eric M Rogers - chapters 12 to 23 (pages 207 to 340) published by Princeton University Press (1960). These pages include astronomy, evidence for a round Earth, evidence for a spinning earth, explanations for many gravitational effects like tides, non-spherical shape of the Earth/precession, variation of 'g' over the Earth's surface. There is also a lot of information on the major contributors over the centuries to our knowledge of gravitation. A brief historical note on these people follows.

Claudius Ptolemy (A.D. 120) assumed the Earth was immovable and tried to explain the strange motion of various stars and planets on that basis. In an enormous book, the "Almagest", he attempted to explain in complex terms the motion of the 'five wandering stars' - the planets.

Nicolaus Copernicus (1510) insisted that the Sun and not the Earth was the centre of the solar system. First to really challenge Ptolemy. He was the first to suggest that the Earth was just another planet. His great work published in 1543, "On the Revolutions of the Heavenly Spheres", had far reaching effects on others working in gravitation.

Tycho Brahe (1580) made very precise and accurate observations of astronomical motions. He did not accept Copernicus' ideas. His excellent data were interpreted by his student Kepler.

Johannes Kepler (1610) Using Tycho Brahe's data he derived three general rules (or laws) for the motion of the planets. He could not explain the rules.

Galileo Galilei (1610) was a great experimenter. He invented the telescope and with it made observations which agreed with Copernicus' ideas. His work caused the first big clash with religious doctrine regarding Earth-centred biblical teaching. His work "Dialogue" was banned and he was imprisoned. (His experiments and scientific method laid the foundations for the study of Mechanics).

Isaac Newton (1680) brought all this together under his theory of Universal Gravitation explaining the moon's motion, the laws of Kepler and the tides, etc. In his mathematical analysis he required calculus - so he invented it as a mathematical tool!

Consideration of Newton's Hypothesis

It is useful to put yourself in Newton's position and examine the hypothesis he put forward for the variation of gravitational force with distance from the Earth. For this you will need the following data on the Earth/moon system (all available to Newton).

Data on the Earth

"g" at the Earth's surface = 9.8 m s^{-2}
radius of the Earth, R_E = 6.4×10^6 m
radius of moon's orbit, r_M = 3.84×10^8 m
period, T, of moon's circular orbit = 27.3 days = 2.36×10^6 s.

take $\frac{R_E}{r_M} = \left[\frac{1}{60}\right]$

Assumptions made by Newton

- All the mass of the Earth may be considered to be concentrated at the centre of the Earth.

- The gravitational attraction of the Earth is what is responsible for the moon's circular motion round the Earth. Thus the observed central acceleration can be calculated from measurements of the moon's motion: $a = \frac{v^2}{r}$.

Hypothesis

Newton asserted that the acceleration due to gravity "g" would quarter if the distance from the centre of the Earth doubles i.e. an inverse square law.

$$\text{"g"} \propto \frac{1}{r^2}$$

- Calculate the central acceleration for the Moon: use $a = \frac{v^2}{R}$ or $a = \frac{4\pi^2 R}{T^2}$ m s^{-2}.
- Compare with the "diluted" gravity at the radius of the Moon's orbit according to the hypothesis, viz. $\frac{1}{(60)^2} \times 9.8 \text{ m s}^{-2}$.

Conclusion

The inverse square law applies to gravitation.

General Data

Planet or satellite	Mass/ kg	Density/ kg m^{-3}	Radius/ m	Grav. accel./ m s^{-2}	Escape velocity/ m s^{-1}	Mean dist from Sun/ m	Mean dist from Earth/ m
Sun	1.99×10^{30}	1.41×10^3	7.0×10^8	274	6.2×10^5	--	1.5×10^{11}
Earth	6.0×10^{24}	5.5×10^3	6.4×10^6	9.8	11.3×10^3	1.5×10^{11}	--
Moon	7.3×10^{22}	3.3×10^3	1.7×10^6	1.6	2.4×10^3	--	3.84×10^8
Mars	6.4×10^{23}	3.9×10^3	3.4×10^6	3.7	5.0×10^3	2.3×10^{11}	--
Venus	4.9×10^{24}	5.3×10^3	6.05×10^6	8.9	10.4×10^3	1.1×10^{11}	--

Inverse Square Law of Gravitation

Newton deduced that this can only be explained if there existed a universal gravitational constant, given the symbol G.

We have already seen that Newton's "hunch" of an inverse square law was correct. It also seems reasonable to assume that the force of gravitation will vary with the masses involved.

$$F \propto m, \quad F \propto M, \quad F \propto \frac{1}{r^2} \qquad \text{giving} \quad F \propto \frac{Mm}{r^2}$$

$$\boxed{F = \frac{GMm}{r^2}} \qquad \text{where } G = 6.67 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$$

Consider the Solar System

$$M \to M_s \quad \text{and} \quad m \to m_p$$

Force of attraction on a planet is: $F = \frac{GM_sm_p}{r^2}$ (r = distance from Sun to planet)

Now consider the central force if we take the motion of the planet to be circular.

Central force $F = \frac{GM_sm_p}{r^2}$

Also $F = \frac{m_pv^2}{r}$ force of gravity supplies the central force.

Thus $\frac{GM_sm_p}{r^2} = \frac{m_pv^2}{r}$ and $v = \frac{2\pi r}{T}$

$$\frac{GM_sm_p}{r^2} = \frac{m_p}{r} \cdot \frac{4\pi^2r^2}{T^2}$$

rearranging $\frac{r^3}{T^2} = \frac{GM_s}{4\pi^2}$

Kepler had already shown that $\frac{r^3}{T^2}$ = a constant, and M_s is a constant, hence it follows that G must be a constant for **all** the planets in the solar system (i.e. a **universal** constant).

Notes:
- We have assumed circular orbits. In reality, orbits are elliptical.
- Remember that Newton's Third Law always applies. The force of gravity is an action-reaction pair. Thus if your weight is 600 N on the Earth; as well as the Earth pulling you down with a force of 600 N, you also pull the Earth up with a force of 600 N.
- Gravitational forces are very weak compared to the electromagnetic force (around 10^{39} times smaller). Electromagnetic forces only come into play when objects are charged or when charges move. These conditions only tend to occur on a relatively small scale. Large objects like the Earth are taken to be electrically neutral.

"Weighing" the Earth

Obtaining a value for "G" allows us to "weigh" the Earth i.e. we can find its mass. Consider the Earth, mass M_e, and an object of mass m on its surface. The gravitational force of attraction can be given by **two** equations:

$$F = mg \qquad \text{and} \qquad F = \frac{GmM_e}{R_e^2}$$

where R_e is the separation of the two masses, i.e. the radius of the earth.

$$\text{Thus} \quad m\,g = \frac{GmM_e}{R_e^2} \qquad M_e = \frac{gR_e^2}{G} = \frac{9.8 \times (6.40 \times 10^6)^2}{6.67 \times 10^{-11}}$$

Thus the mass of the Earth $= 6.02 \times 10^{24}$ kg

The Gravitational Field

In earlier work on gravity we restricted the study of gravity to small height variations near the earth's surface where the force of gravity could be considered constant.

Thus $F_{grav} = m\,g$

Also $E_p = m\,g\,h$ where g = constant ($9.8\ N\ kg^{-1}$)

When considering the Earth-Moon System or the Solar System we cannot restrict our discussions to small distance variations. When we consider force and energy changes on a large scale we have to take into account the variation of force with distance.

Definition of Gravitational Field at a point.

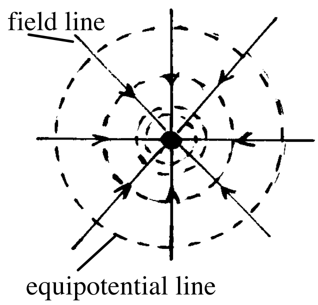
This is defined to be the force per unit mass at the point. i.e. $g = \frac{F}{m}$

The concept of a field was not used in Newton's time. Fields were introduced by Faraday in his work on electromagnetism and only later applied to gravity.

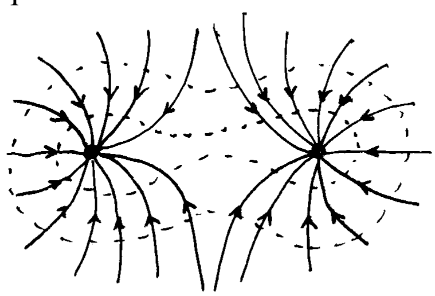
Note that g and F above are both vectors and whenever forces or fields are added this must be done vectorially.

Field Patterns (and Equipotential Lines)

(i) An Isolated 'Point' Mass



(ii) Two Equal 'Point' Masses



(equipotential lines shown broken)

Note that equipotential lines are always at **right angles** to field lines.

Variation of g with Height above the Earth (and inside the Earth)

An object of mass m is on the surface of the Earth (mass M). We now know that the weight of the mass can be expressed using Universal Gravitation.

Thus $mg = \frac{GMm}{r^2}$ (r = radius of Earth in this case)

$g = \frac{GM}{r^2}$ (note that $g \alpha \frac{1}{r^2}$ above the Earth's surface)

However the density of the Earth is **not** uniform and this causes an unusual variation of g with radii **inside** the Earth.

Gravitational Potential

We define the gravitational potential (V_p) at a point in a gravitational field to be the work done by external forces in moving unit mass m from infinity to that point.

$$V_p = \frac{\text{work done}}{\text{mass}}$$

We define the theoretical zero of gravitational potential for an isolated point mass to be at **infinity**. (Sometimes it is convenient to treat the surface of the Earth as the practical zero of potential. This is valid when we are dealing with **differences** in potential.)

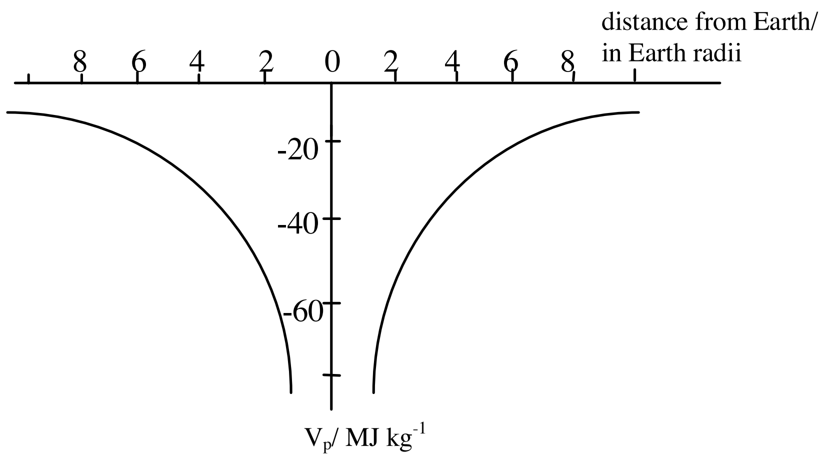
Gravitational Potential at a distance r from mass m

This is given by the equation below.

$$\boxed{\text{gravitational potential } V_P = -\frac{GM}{r}}$$

unit of V_P: J kg^{-1}

The Gravitational Potential 'Well' of the Earth

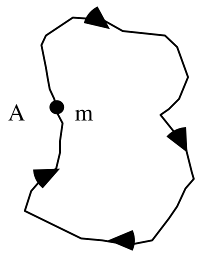


This graph gives an indication of how masses are 'trapped' in the Earth's field.

Conservative field

The force of gravity is known as a conservative force because the work done by the force on a particle that moves through any round trip is zero i.e. energy is conserved. For example if a ball is thrown vertically upwards, it will, if we assume air resistance to be negligible, return to the thrower's hand with the same kinetic energy that it had when it left the hand.

An unusual consequence of this situation can be illustrated by considering the following path taken in moving mass m on a round trip from point A in the Earth's gravitational field. If we assume that the only force acting is the force of gravity and that this acts vertically downward, work is done **only** when the mass is moving vertically, i.e. only vertical components of the displacement need be considered. Thus for the path shown below the work done is zero.



The overall energy required for this round trip is zero.

By this argument a non conservative force is one which causes the energy of the system to change e.g. friction causes a decrease in the kinetic energy. Air resistance or surface friction can become significant and friction is therefore labelled as a non conservative force.

Escape Velocity

The escape velocity for a mass m escaping to infinity from a point in a gravitational field is the **minimum velocity** the mass must have which would allow it to escape the gravitational field.

At the surface of a planet the gravitational potential is given by: $V = -\frac{GM}{r}$.

The **potential energy** of mass m is given by **V x m** (from the definition of gravitational potential).

$$E_p = -\frac{GMm}{r}$$

The potential energy of the mass at infinity is zero. Therefore to escape completely from the sphere the mass must be given energy equivalent in size to $\frac{GMm}{r}$.

To escape completely, the mass must just reach infinity with its $E_k = 0$

(Note that the condition for this is that at all points; $E_k + E_p = 0$).

at the surface of the planet $\frac{1}{2} m v_e^2 - \frac{GMm}{r} = 0$ m cancels

$$v_e^2 = \frac{2GM}{r}$$

$$\boxed{v_e = \sqrt{\frac{2GM}{r}}}$$ or greater

Atmospheric Consequences:

$v_{r.m.s.}$ of H_2 molecules = 1.9 km s^{-1} (at 0°C)

$v_{r.m.s.}$ of O_2 molecules = 0.5 km s^{-1} (at 0°C)

When we consider the range of molecular speeds for hydrogen molecules it is not surprising to find that the rate of loss to outer space is considerable. In fact there is very little hydrogen remaining in the atmosphere. Oxygen molecules on the other hand simply have too small a velocity to escape the pull of the Earth.
The Moon has no atmosphere because the escape velocity (2.4 km s^{-1}) is so small that any gaseous molecules will have enough energy to escape from the moon.

Black Holes and Photons in a Gravitational Field

A dense star with a sufficiently large mass/small radius could have an escape velocity greater than 3 x 10^8 m s^{-1}. This means that light emitted from its surface could not escape - hence the name **black hole**.

The physics of the black hole cannot be explained using Newton's Theory. The correct theory was described by Einstein in his General Theory of Relativity (1915). Another physicist called Schwarzschild calculated the radius of a spherical mass from which light cannot escape. It is given here for interest only $r = \frac{2GM}{c^2}$.

Photons are affected by a gravitational field. There is gravitational force of attraction on the photon. Thus photons passing a massive star are **deflected** by that star and stellar objects 'behind' the star appear at a very slightly different position because of the bending of the photon's path.

Further Discussion on Black Holes

If a small rocket is fired vertically upwards from the surface of a planet, the velocity of the rocket decreases as the initial kinetic energy is changed to gravitational potential energy. Eventually the rocket comes to rest, retraces its path downwards and reaches an observer near to the launch pad.

Now consider what happens when a photon is emitted from the surface of a star of radius r and mass M. The energy of the photon, hf, decreases as it travels to positions of greater gravitational potential energy but **the velocity of the photon remains the same.** Observers at different heights will observe the frequency and hence the wavelength of the photon changing, i.e. blue light emitted from the surface would be observed as red light at a distance from a sufficiently massive, high density star. (N.B. this is known as the gravitational redshift **not** the well known Doppler redshift caused by the expanding universe).

If the mass and density of the body are greater than certain critical values, the frequency of the photon will decrease to zero at a finite distance from the surface and the photon will not be observed at greater distances.

It may be of interest to you to know that the Sun is not massive enough to become a black hole. The critical mass is around 3 times the mass of our Sun.

Satellites in Circular Orbit

This is a very important application of gravitation.
The central force required to keep the satellite in orbit is provided by the force of gravity.

Thus: $$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v = \sqrt{\frac{GM}{r}} \qquad \text{but} \qquad v = \frac{2\pi r}{T}$$

$$T = 2\pi r\sqrt{\frac{r}{GM}} = 2\pi\sqrt{\frac{r^3}{GM}}$$

Thus a satellite orbiting the Earth at radius, r, has an orbit period, $T = 2\pi\sqrt{\frac{r^3}{GM}}$

Energy and Satellite Motion

Consider a satellite of mass m a distance r from the centre of the parent planet of mass M where $M >> m$.

Since $$\frac{GMm}{r^2} = m\frac{v^2}{r}$$

Re-arranging, we get $$\frac{1}{2}mv^2 = \frac{GMm}{2r} \quad ; \quad \text{thus} \quad E_k = \frac{GMm}{2r}$$

Note that E_k is always positive.

But the gravitational potential energy of the system, $E_p = -\frac{GMm}{r}$

Note that E_p is always negative.

Thus the total energy is $$E_{tot} = E_k + E_p$$

$$= \frac{GMm}{2r} + \left[-\frac{GMm}{r}\right]$$

$$E_{tot} = -\frac{GMm}{2r}$$

Care has to be taken when calculating the energy required to move satellites from one orbit to another to remember to include **both** changes in gravitational potential energy and changes in kinetic energy.

Some Consequences of Gravitational Fields

The notes which follow are included as **illustrations** of the previous theory.

Kepler's Laws

Applied to the Solar System these laws are as follows:

- The planets move in elliptical orbits with the Sun at one focus,
- The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.
- The square of the orbital period of a planet is proportional to the cube of the semi-major axis of the orbit.

Tides

The two tides per day that we observe are caused by the **unequal attractions** of the Moon (and Sun) for masses at different sides of the Earth. In addition the rotation of the Earth and Moon also has an effect on tidal patterns.

The Sun causes two tides per day and the Moon causes two tides every 25 hours. When these tides are in phase (i.e. acting together) **spring** tides are produced. When these tides are out of phase **neap** tides are produced. Spring tides are therefore larger than neap tides. The tidal humps are held 'stationary' by the attraction of the Moon and the earth rotates beneath them. Note that, due to tidal friction and inertia, there is a time lag for tides i.e. the tide is not directly 'below' the Moon. In most places tides arrive around 6 hours late.

Variation of "g" over the Earth's Surface

The greatest value for "g" at sea level is found at the poles and the least value is found at the equator. This is caused by the rotation of the earth.

Masses at the equator experience the maximum spin of the earth. These masses are in circular motion with a period of 24 hours at a radius of 6400 km. Thus, part of a mass's weight has to be used to supply the small central force due to this circular motion. This causes the measured value of "g" to be smaller.

Calculation of central acceleration at the equator:

$$a = \frac{v^2}{r} \quad \text{and} \quad v = \frac{2\pi r}{T} \quad \text{giving} \quad a = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 \times 6.4 \times 10^6}{(24 \times 60 \times 60)^2} = 0.034 \text{ m s}^{-2}$$

Observed values for "g": at poles = 9.832 m s^{-2} and at equator = 9.780 m s^{-2}

difference is 0.052 m s^{-2}

Most of the difference has been accounted for. The remaining 0.018 m s^{-2} is due to the non-spherical shape of the Earth. The equatorial radius exceeds the polar radius by 21 km. This flattening at the poles has been caused by the centrifuge effect on the liquid Earth as it cools. The Earth is 4600 million years old and is still cooling down. The poles nearer the centre of the Earth than the equator experience a greater pull.

In Scotland "g" lies between these two extremes at around 9.81 or 9.82 m s^{-2}. Locally "g" varies depending on the underlying rocks/sediments. Geologists use this fact to take *gravimetric surveys* before drilling. The shape of underlying strata can often be deduced from the variation of "g" over the area being surveyed. Obviously very accurate means of measuring "g" are required.