UNCERTAINTIES

Summary of the Basic Theory associated with Uncertainty

It is important to realise that whenever a physical quantity is being measured there will always be a degree of inaccuracy associated with the measurement. Thus, whenever experimental measurements are made these inaccuracies or **uncertainties** should be estimated.

Calibration Uncertainty

All measuring instruments have an associated inaccuracy known as the calibration uncertainty. For instance when a wooden metre stick is used to measure a length in the laboratory it is a fair estimate that the metre length of wood itself will be accurate to within 0.5 mm. The table below gives some typical examples of calibration uncertainties:

Instrument	**Calibration Uncertainty**
Metre Stick (wood)	0.5 mm
Ruler made of Steel	0.1 mm
Digital Meter	0.5% of reading + 1 in last digit

Thus for an ammeter reading (from a digital meter) of 3.54 A the uncertainty will be:

$$(0.5\% \text{ of } 3.54 \text{ A}) + 0.01 = 0.018 + 0.01 = 0.02 + 0.01 = 0.03 \text{ A}$$

Thus final value of current should be quoted as: current = 3.54 ± 0.03 A.

Systematic Effects

As the name suggests, uncertainties can arise because of the system used to gather the information. The measurement of time is a good example of this. If you were using a stopwatch which after much use now runs slow, the uncertainty in its use may in fact be worse than its calibration uncertainty. This effect would only be detected by using an independent instrument to check the stop watch. Similarly if a student consistently measured the oscillation of a pendulum wrongly e.g. started the stopwatch at the wrong point in the first swing, then the period of the pendulum would have a systematic uncertainty. This uncertainty can be detected if several different numbers of swings are timed and T is plotted against $\sqrt{l}$. The graph will not pass through the origin as it should, if the experiment had been carried out properly.

Scale Reading Uncertainty

This value indicates how well an instrument scale can be read.
An estimate of the reading uncertainty for an analogue scale is taken as ± half the smallest scale division. For very widely spaced scales a reasonable estimate should be made. For a digital scale, the reading uncertainty is taken as ± 1 in the least significant digit. This has been mentioned above under calibration uncertainty.

Random Uncertainties

It is always advisable to repeat measurements if it is possible. This allows us to check that nothing has gone wrong in taking the first measurement. We usually find that there is a spread of values for the quantity being measured and the random uncertainty in the measurements can be determined from this spread.

Mean and random uncertainty in the mean

The **mean** of a number, n, of measurements of quantity P is found in the usual way:

$$P_{mean} = \frac{\text{sum of all the measurements}}{\text{number of measurements}} \qquad (P_{mean} = \frac{\sum P_i}{n}).$$

The **approximate random uncertainty** in the mean is found from:

$$\text{uncertainty in P} = \frac{P_{maximum} - P_{minimum}}{n}$$

This method is suitable if we have more than about five readings.

Combining Uncertainties

Addition and Subtraction

When two quantities, A and B, are added (or subtracted) the uncertainty (ΔS) in the sum (or difference) is given by:

$$\Delta S = \sqrt{(\Delta A)^2 + (\Delta B)^2}$$

where ΔA is the uncertainty in A and ΔB is the uncertainty in B.

Thus subtracting two quantities which are nearly the same can result in very high percentage uncertainty.

Multiplication and Division

When two quantities, A and B, are multiplied or divided the **fractional uncertainties** are important. Thus if P = A x B or if $P = \frac{A}{B}$

$$\frac{\Delta P}{P} = \sqrt{[\frac{\Delta A}{A}]^2 + [\frac{\Delta B}{B}]^2}$$

This must also apply to **percentage** uncertainties.

$$\%\text{ uncertainty in P} = \sqrt{(\%\text{ uncertainty in A})^2 + (\%\text{ uncertainty in B})^2}$$

Powers

If $P = A^n$ then: % uncertainty in P = n x % uncertainty in A.

For example, if the % uncertainty in a distance s is 1.5% and our formula involved s^2 then the % uncertainty in s^2 would be 3%.

Graphs

Individual points should include 'error bars' where appropriate. These are used to enable the best straight line or curve to be drawn.

When plotting a straight line graph it is possible to get the uncertainty in the gradient by employing the "centroid" method. This involves finding the maximum and minimum gradients from the scatter of points which make up the graph.

First the centroid is found. This is the mean of all the x co-ordinates and the mean of all the y co-ordinates. The best straight line is drawn through this centroid. A top line is then drawn, parallel to this best line, so that it passes through the point (not its error bar) that lies furthest above the best line. A similar line is drawn below, to give a parallelogram. The gradients of the two diagonals of this parallelogram, the 'worst lines', are then calculated. Let these be m_1 and m_2.

The uncertainty in the gradient is given by: $\Delta m = \dfrac{m_1 - m_2}{2\sqrt{(n-2)}}$

where n is the number of points on the graph (excluding the centroid).

The uncertainty in the intercept is found by noting where the two 'worst lines' cut the y axis. Let these be c_1 and c_2.

The uncertainty in the intercept is given by: $\Delta c = \dfrac{c_1 - c_2}{2\sqrt{(n-2)}}$

Dominant Uncertainty in an Expression

Consider three quantities multiplied together in an expression. If one quantity has a much larger percentage uncertainty than the other two, then this largest uncertainty can be applied to the quantity in question.

For example, if in equation $E_p = m\,g\,h$, the % uncertainty in m is 1%; the % uncertainty in g is 1% and the % uncertainty in h is 5% then we can safely say that the % uncertainty in E_p is 5%.

Check: % uncertainty in $E_p = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27} = 5.2\%$.

Thus, taking 5% as the overall estimate of uncertainty is a statistically acceptable approximation as long as the dominant uncertainty is considerably more than the other uncertainties. As a general rule a dominant uncertainty should be three times any other uncertainty. Thus if an uncertainty is less than a third of another uncertainty it can be neglected.

Comparing Results of Experiments

If we arrive at a numerical result in the form $x \pm \Delta x$, this allows us to compare the results for other experiments measuring the same quantity. Doing this may allow us to evaluate how successful or otherwise the method has been.

A good example of this is an analysis of two different methods of measuring g, the acceleration due to gravity.

Method A - pendulum	result: $g = 9.5 \pm 0.4$ m s^{-2}
Method B - oscillating spring	result: $g = 9.82 \pm 0.09$ m s^{-2}

Both of these values lie within the accepted value for g in Scotland which is between 9.815 m s^{-2} and 9.819 m s^{-2}. However we can say that the pendulum method is obviously more inaccurate but nevertheless still a valid measure. If the value had been 9.5 ± 0.2 m s^{-2} then this would have indicated that the method used could have been improved since it lies outside the accepted value for g. A repeat of the measurements should be carried out.

TUTORIAL

Uncertainties

1 Three packages have to be added to the payload of the Space Shuttle. Their masses have been measured as follows:
m_1 = (112 ± 1) kg m_2 = (252 ± 2) kg   and m_3 = (151 ± 1) kg.
Calculate the total mass to be added and the uncertainty in the total.

2 When using a travelling microscope the following measurements were made.
Reading 1 = (112.1 ± 0.2) mm   Reading 2 = (114.5 ± 0.2) mm.
Calculate:
(a) the percentage uncertainty in the sum of these readings
(b) the percentage uncertainty in the difference of these readings
(c) Which of these, sum or difference, is usually needed for the travelling microscope?

3 A block of building material has been carefully machined to undergo tests. Its dimensions and mass are as follows:

length = 0.050 ± 0.001 m
breadth = 0.100 ± 0.001 m
height = 0.040 ± 0.001 m
mass = 0.560 ± 0.002 kg

(a) From this data, calculate the density of this material.
(b) Find the uncertainty in this value of density and express it as a percentage.

4 The radius of a sphere is measured to be (1.2 ± 0.1) x 10^{-2} m.
If the volume of a sphere is given as $\frac{4}{3}\pi r^3$, where r is the radius of the sphere, calculate the volume of the sphere, quoting the uncertainty in your answer.

5 A uniform disc is to be used as a flywheel in a new design of small engine. Its moment of inertia has to be known. The following method is used:

The diameter of the disc is measured with a metre stick at 8 different positions round the rim and its mass is measured on a balance which was accurate to 10 g.

Diameters	0.245 m	0.249 m	0.255 m	0.248 m
	0.243 m	0.247 m	0.251 m	0.246 m

Mass 4.04 kg

Use the formula for the moment of inertia = $\frac{1}{2}MR^2$, where R is the radius of the disc. Find the moment of inertia, quoting a value for the uncertainty associated with your answer.

6 Calculate the refractive index of a glass block from the following information:
Angle of incidence = (46 ± 1)°   Angle of refraction = (28 ± 1)°.
Make sure you quote an uncertainty in your answer.